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FOREST MANAGEMENT: STATIC AND DYNAMIC MODELS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled FOREST MANAGEMENT: STATIC AND DYNAMIC MODELS submitted by WILLIAM G. HOWARD in partial fulfilment of the requirements for the degree of Master of Arts.

ABSTRACT

The determination of the harvest schedule over time is a fundamental problem of forest management. Traditionally, static analysis is used to determine the age at which trees should be cut in the steady state. Dynamic analysis is then used to determine that sequence of harvests which will result in this steady state.

This study attempts to review and present both static and dynamic models to illustrate the economic solution to this problem. The static problem is the determination of the age at which trees should be harvested in order to maximize discounted net revenues. In addition to the traditional solution, two static economic models are presented. The first considers the age at which trees should be harvested when only one crop is anticipated while the second model considers the case where multiple crops can be realized from an area of land.

Several economic models of dynamic resource exploitation are reviewed. The nature of the possible steady states are examined. These steady states are contrasted with the steady states that result under the static treatment of the problem.

Two problems with dynamic models of resource exploitation as they pertain to the growth of forests are discussed. Both are attributable to the simplifications of the growth process which the models assume. In some cases and with certain modifications a realistic model of forest growth can be constructed.

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CONTENTS

CHAPTER ONE

INTRODUCTION

This study examines the theory of the economics of forest management under both a comparative static and dynamic framework. In the former area, a well established model of forest investment exists. This model will be developed and the important results summarized. Very little attention has been devoted to dynamic models of forestry, however a significant amount of applicable research on natural resources in general exists. Three models in this area, two of which purport to be applicable, will be reviewed and critically evaluated.

Natural Resources

The role which natural resources play in an economy should be distinguished at the outset. According to Ciriacy-Wantrup:

> . . . one may differentiate three broad classes of resources: natural, cultural and human. This trinity corresponds to the trinity of "factors" of production . . .[1]

As one of three classes of resources, natural resources are often characterized by their origin. They may be viewed as a gift from nature, to be distinguished from capital goods which are produced through the interaction of cultural and human resources. However, as Scott points out, this is not to say that resources cannot be regarded as part of the social capital.[2]

One of the immeasurable contributions Professor Ciriacy-Wantrup has made to natural resource economic thought lies in the classification of these resources.[3] This system of classification, based on the inherent characteristics of the resources, provides the insight necessary for policy discussions.

Natural resources may be classed as renewable or non-renewable. In the latter case, resources are viewed as a stock which is not augmentable with present technology. The stock may or may not significantly decay or deteriorate over time and this fact allows a further subdivision within the non-renewable class. Renewable resources are essentially stock resources from which a flow emanates. The body of of the sun may be considered as a stock from which a flow of solar radiation originates. A forest may be considered as a stock, the growth of which constitutes a flow. These examples serve to introduce a distinction among renewable resources. In some instances

(forests), the flow may be stored while in other cases (sunlight) this is not possible. Although this approach would seem fruitful in distinguishing classes of renewable resources, problems arise in its application.[4] Ciriacy-Wantrup distinguishes two classes of renewable resources depending upon whether human action does or does not significantly affect the flow. Forests are obviously an example of the first category while sunlight falls into the second. The renewable resources significantly affected by the actions of man can be further subdivided into those cases which are subject to exhaustion and those which are not. Care should be exercised in the use of the term exhaustion.

Forests and fisheries are renewable resources which may be significantly affected by the actions of man. Nothing of a general nature may be said about the exhaustion of these resources. Certain species of trees are biologically exhaustible while others may thrive under nearly all conceivable actions of man. Fisheries are biologically exhaustible. Biological exhaustion due to exploitation is almost inconceivable in all but the rare case since one would expect costs to vary inversely with the stock of the resource. Biological and economic exhaustion are therefore distinguished.[5]

Common Property

Since Gordon's article on the economics of a common property natural resource[6] much attention has been focused on the problem. Historically, forests were at one time subject to the problem as land tenure systems were absent or weak. The resource was systematically exploited until depletion was clearly recognized. Tenure systems developed as a solution to the problem. Management of the resource appeared and undoubtably one of the first questions posed by those concerned was how the resource should be managed. With timber scarcity the focus of attention, the concept of maximizing the physical yield from the land developed.

Without tenure systems, the individual exploiting the resource places no value on the stock of the resource. Since the flow emanates from the stock, systematic depletion of the stock reduced the productive capacity of the resource. The problem was essentially solved for forests with the introduction of tenure systems. The establishment of property rights is impractical in the case of the fisheries, however. It will be seen that a system of quotas or prices may be used to direct the exploitation of the resource optimally.

The key to management of the common property

resource is therefore the valuation of the stock of the resource. Establishment of land tenure systems in forestry allows the individual to value the stock through the rights of exclusion. The maintenance of the productive capacity of the resource allows the individual to benefit in the future. Where tenure systems are not feasible, a system of quotas may assure the conservation of the resource. Each individual exploiting the resource is excluded from harvesting more than his quota. An alternative to quotas may be prices. These prices may vary over time and in an economic manner exclude the users of the resource from depleting the resource through excessive harvesting. In all cases, the stock of the resource is valued whether indirectly through tenure or directly through quotas or prices.

Forests and Time

Time plays an important and somewhat unique role in the management of forests. The maturation of a single crop may take from a few decades to over a century. Once established, the crop grows through time. The growth in one time period constitutes the flow from the stock of the resource. This growth is then stored as part of the stock of the resource. The volume on an area of forested land therefore represents the cumulative stored growth since the

establishment of the crop.

If the goal of management is to maximize the physical yield from forested land a first proposal to achieve that objective might be that the growth or increment be cut or harvested when it is greatest. Stated in another manner, one might suggest that the forest be allowed to grow until the growth is maximum and at this point establish a steady state with the level of harvest equal to the growth in that time period. This, however, ignores the fact that the separation of the growth from the stock is not possible:

> The increment of the stand is never cut. All that one can do is to cut entire trees that in their total volume *represent* the increment of the stand.[7]

At the risk of being repetitive, this point can be illustrated through two simple examples. First, assuming that the crop consists of one tree and realizing that the growth cannot be separated from the tree, when should the tree be harvested to maximize the physical yield? Of course, once the tree has been harvested another will be established in its place. If the age of the tree is in terms of years, the answer is at that age when the average annual yield is maximum.[8] A more realistic problem might be stated in the same terms with "stand" or "acre" replacing "tree".[9] To maximize the physical yield the

acre or stand would be harvested when the average annual yield is maximized. The average annual yield for any age is therefore a steady state point.

Organization of the Study

This study consists of four chapters. Chapter Two will present the technology and two comparative static models. The first model considers the question of when an acre should be harvested and what amount of labor services employed to plant the land given all prices. Although the actual formulation of the problem is slightly different than that of previous authors, the results are simple and economically appealing. The second model is quite similar with one exception. Recognizing the fact that after harvesting a new crop will be established the same question is again asked. This model will be considered in slightly more detail. The second chapter will also introduce the concept of maximum sustained yield.

Chapter Three will review three dynamic models of resource exploitation. Plourde's[10] model assumes that utility is a function of the harvest of the resource at any point in time. If appropriately discounted intertemporal utilities may be added to yield a measure of welfare over time, the question of the optimal rate of harvest of the resource may be investigated. The second model,

developed by Smith,[11] assumes that the rate of exploitation of the resource depends on the number of units of capital employed and the rate at which each unit produces output. The third model, attributable to Brown,[12] assumes the agency controlling the resource wishes to maximize the discounted value of net revenues over time and the exploitation of the resource over time is investigated. Although these models are in large directed towards the common property renewable resource such as fisheries they are still of interest in the case of an appropriated resource. Particularly, the conditions under which a steady state results and the nature of the steady state are valuable.

Chapter Four will examine the models under a more critical framework. Specifically, the biological constraint will be examined as well as the nature of the steady-states possible. Two problems which arise from the misspecification of the biological constraint will be discussed. Finally, Chapter Four will conclude the study.

Footnotes

[1]Ciriacy-Wantrup, S. V., *Resource Conservation, Economics and Policies*, Third Edition (Berkeley: University of California Press, 1963), p. 29.

[2]Scott, A., *Natural Resources* (Toronto: McClelland and Stewart Ltd., 1973), p. 16.

[3]*Op. Cit.*, p. 42.

[4]Some resources with or without storeable flows are exhaustible. A simple classification system is therefore whether or not the flow is significantly affected by human action since only those resources significantly affected are exhaustible.

[5]A similar distinction is made in a footnote in:
Smith, V. L., "Economics of Production From Natural Resources," *A.E.R.* 58 (1968):409-431, p. 409.

[6]Gordon, H. S., "The Economic Theory of a Common Property Resource: The Fishery," *J.P.E.* 62 (1954):124-142.

[7]Davis, K. P., *Forest Management: Regulation and Valuation*, Second Edition (New York: McGraw-Hill, 1966), p. 179.

[8]If more than two ages give maximum average annual yield obviously the smaller age is preferred.

[9]To make the transition in the text between "tree" and "acre" in the example requires that clearcutting be the only feasible harvest technique and that all trees are the same age on the acre.

[10]Plourde, C. G., "A Simple Model of Replenishable Natural Resource Exploitation," *A.E.R.* 50 (1960):518-522.

[11]Smith, V., "Economics of Production from Natural Resources," *A.E.R.* 58 (1968):409-431.

[12]Brown, G., "An Optimal Program for Managing Common Property Resources with Congestion Externalities," *J.P.E.* 82 (1974): 163-173.

CHAPTER TWO

WEALTH MAXIMIZATION

The purpose of this chapter is to introduce the technology and review a comparative static model of wealth maximization. The entrepreneur is assumed to maximize wealth rather than profits since decisions made in one time period will influence the value of net revenues (discounted) of future time periods. Wealth is therefore defined as the discounted value of all future net revenues.

For a given area of land, the entrepreneur plans to establish a cycle, or rotation which will continually utilize the land. A crop of trees will be planted initially on this land and will then be allowed to grow or mature for a number of years. The crop will then be harvested and the land subsequently replanted to begin another cycle. Following tradition we will call this cycle a rotation and the length of time between harvests the rotation length.

It will be assumed that the entrepreneur operates in competitive markets and that all parameters are constant

through time. Therefore, the entrepreneur has only to determine the rotation length and the amount of labor services required to plant the crop. With constant parameters assumed, once determined, the value these variables assume will be fixed over time and a sequence of identical rotations will be established.

Under these assumptions, profit and wealth maximization can be clearly distinguished. If the entrepreneur decides to delay the harvest of the first crop or rotation one year, then all future harvests are similarly delayed. This decision obviously alters the value of discounted net revenues from future harvests. Therefore, it may be concluded that wealth maximization or, alternatively, maximization of the value of all discounted net revenues is most realistic.

The rotation length is a choice parameter for the entrepreneur. This does not imply that time is a factor of production but rather it implies that the entrepreneur controls the length of time over which production takes place. Since it is assumed that labor services are responsible for the establishment of the crop and a period of time elapses between this application and maturation of the trees; production is of a point-input, point-output type.[1] Another interesting feature of this process is that the entrepreneur incurs costs many periods before reveneus are achieved[2], and for this reason he must have a wage fund with which to pay for the initial services of labor.

The Technology

The production process is a point-input, point-output type, with land and labor as the factor inputs and the length of the production period under control.[3] Rather than simply defining the production function, it may be useful to quote Wicksell's thoughts on a rather similar problem:

> In an individual firm, the value W of the finished product available during a given unit of time (say one year) would clearly be a function of the quantities of labor and land employed (a and b) and also of the periods of time (t and τ) for which each was invested in production:--

$$W = f(a,b,t,\tau).^{4} \qquad \ldots (2.10)$$

Equation (2.10) may be used to establish a production function for the purposes at hand. It first should be noted that Wicksell considered the value of the finished product however his implied production function must obviously incorporate the same arguments. It has already been indicated that land and labor are the factors of production in forestry and that the process begins with the application of labor. Therefore, denoting the physical output volume as $\overline{V}$, equation (2.10) may be altered:

$$\overline{V} = g(a,b,0,\tau) \qquad \ldots (2.11)$$

Assuming that the production function is linearly homogeneous in land and labor and that land services are (for example) denoted in acre terms, (2.11) can be rewritten:

$$\frac{\bar{V}}{b} = g\left(\frac{a}{b}, 1, 0, \tau\right) = F\left(\frac{a}{b} \equiv L, \tau\right) \qquad . . . (2.12)$$

To explicitly note that τ is a controlled parameter and that for the purposes at hand volume per acre will be written as V, (2.12) will always be written:

$$V = F(L;\tau) \qquad . . . (2.13)$$

Simply stated, (2.13) reveals that with a given input of labor per acre L_1, and after τ_1 periods measured from the time of labor application, $V_1 = F(L_1;\tau_1)$ units of volume of wood will have been produced on one acre. Therefore, τ_1 is the rotation length.

Wicksell also considered the case where "the grapes themselves were a gift of nature"[5]. In terms of (2.13), this may be written:

$$V = F(0;\tau) = G(\tau) \qquad . . . (2.14)$$

This equation is the form used as the growth function in

forestry. Labor is not viewed as a factor in production although labor plays a key role. The properties which this function will most often exhibit are given by Mac-Kinney *et al:*

> 1. The curve is limited between zero yield at inception and a finite maximum yield at advanced age . . . .
> 2. The curve exhibits a declining rate of percentage increase . . . .
> 3. The slope of the curve increases with increasing yield in early life and decreases with increasing yield in later life . . . .[6]

The properties lead the authors to propose:

$$\log_e V = \alpha - \frac{\beta}{\tau} \qquad \ldots (2.15)$$

as an acceptable form.[7] Due to certain properties of the logistic curve, the authors reject it as unacceptable for use in forestry.[8]

Volume functions may be specific to a given species and environment. There is no universal form for the function and equation (2.15) is introduced only as a form which illustrates the expected characteristics.

Assuming positive but diminishing returns to the factor inputs, the technology may be summarized:

$$\frac{\partial F(L;\tau)}{\partial L} > 0 \quad , \quad \frac{\partial F(L;\tau)}{\partial \tau} > 0$$

$$\frac{\partial^2 F(L;\tau)}{\partial L^2} < 0 \quad , \quad \frac{\partial^2 F(L;\tau)}{\partial \tau^2} \gtrless 0 \text{ as } \tau \lessgtr \tau'$$

and:[9]

$$\begin{aligned} F(L;0) &= 0 \\ F(L;\infty) &= \alpha,\ 0<\alpha<\infty \\ F(0;\tau) &= G(\tau) \\ F(\infty;\tau) &= H(\tau) \end{aligned}$$

In this model, increasing the labor input is assumed to increase the number and therefore the density of trees. As the density increases, intraspecific competition intensifies and diminishing returns result.

Maximum Sustained Yield and the Rotation Length

The rotation length in forestry is usually determined independently of economic considerations. Traditionally, the rotation length for a single acre is found by maximizing the average annual yield[10] or:

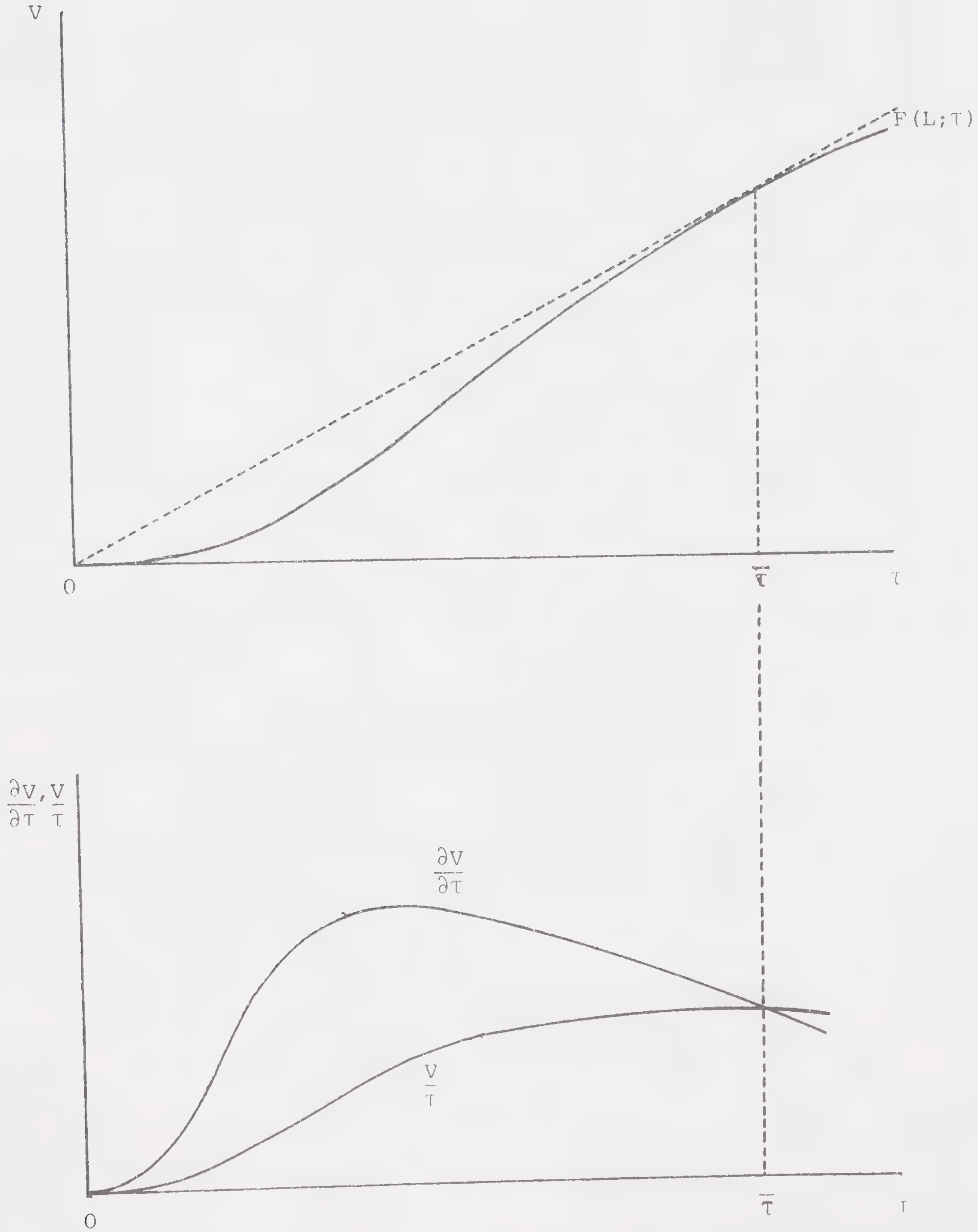
$$\underset{\tau}{\mathrm{MAX}} \left\{ \frac{F(L;\tau)}{\tau} \right\} \qquad . . . (2.16)$$

This maximization leads to the foresters' rule that the mean annual increment must equal the periodic annual increment or, more precisely:

$$\frac{F(L;\bar{\tau})}{\bar{\tau}} = \left. \frac{\partial F(L;\tau)}{\partial \tau} \right|_{\tau=\bar{\tau}} \qquad . . . (2.17)$$

See Fig. 2.1.

Figure 2.1


V
F(L;τ)
0
$\bar{\tau}$
τ
$\frac{\partial V}{\partial \tau}, \frac{V}{\tau}$
$\frac{\partial V}{\partial \tau}$
$\frac{V}{\tau}$
0
$\bar{\tau}$
τ

For a whole forest, every acre growing according to (2.13), the rotation length is the same for each acre. Therefore, under this rule, every acre would be harvested at the same age.

Maximum sustained yield is the goal of many forest managers. Maximum yield is achieved if every acre is harvested at that τ (age) determined by (2.16). Sustained yield management implies a steady flow of output over time from the forest.[11] Maximum sustained yield is a concept which embodies both the ideals of maximum yield and sustained yield. The nature of a maximum sustained yield forest is unintentionally described by Metzler while using trees as an example of a capital good:

> . . . he will have to let his trees attain an age of t_0 before harvesting. Suppose t_0 is 35 years. In order to produce an annual output of W_0 per worker, the business man will have to have 35 wood-lots with a uniform age distribution of trees varying between one and 35 years. In this way, one of his woodlots will be ready for cutting and replanting every year.[12]

A forest with a uniform age distribution has traditionally been called a regulated forest.[13] When each acre is harvested according to (2.17) and the uniform age distribution exists, maximum sustained yield has been achieved.

The rotation length (t_0) determined by Metzler does not result from (2.16). Economic considerations must

be introduced to illustrate the solution.

Profit Maximization: The Single Rotation

It is assumed here that the entrepreneur maximizes profits for a single rotation. The rate of interest (ρ), the nominal wage (ω) and the nominal sale price of standing timber (p) are given and constant over time. All markets are assumed to be competitive.

This simple problem and solution are not novel and an appeal to the literature will provide the solutions. Assuming the technology is given by (2.14), Böhm-Baverk's solution may be quoted:

> Economic profitableness obviously accompanies a lengthening of the waiting period only as long as the yearly growth of the tree represents a greater percentage of the wood of the tree than the prevailing interest rate.[14]

Wicksell arrives at the same solution,[15] which, in terms of (2.14) is:

$$\rho = \frac{dG(\tau)/d\tau}{G(\tau)} \qquad \text{. . . (2.18)}$$

By assuming the technology is given by (2.13), this problem can be written:

$$\underset{L,\tau}{\text{MAX}}\left\{pF(L;\tau)e^{-\rho\tau} - \omega L\right\} \qquad \ldots (2.19)$$

As stated previously, it is assumed that wages are paid when the trees are planted and revenues received when the trees are mature. The maximization yields:

$$\frac{\partial F(L;\tau)/\partial\tau}{F(L;\tau)} = \rho \qquad \ldots (2.20)$$

$$p\,\frac{\partial F(L;\tau)e^{-\rho\tau}}{\partial L} = \omega \qquad \ldots (2.21)$$

The reader will note that (2.20) is identical to the Böhm-Baverk, Wicksell solution (2.18) while (2.21) indicates that the discounted value of the marginal product will equal the wage rate.

Recalling that the land will yield a perpetual series of harvests, (2.19) should be accordingly modified. The entrepreneur should therefore maximize wealth, or alternatively, the value of land.

Wealth Maximization: Rotations in Perpetuity

Wicksell and Böhm-Baverk considered trees as an example of a capital good. Their investigations were not aimed towards developing a realistic model of forest

investment.

The first model which assumed rotations in perpetuity was developed by Faustman.[16] His investigations were directed towards determining the value of land under forest cultivation. Since Faustman, the solution has been reaffirmed by many authors, including Gaffney,[17] Hirshleifer[18] and Samuelson.[19]

Faustman assumed a yield function invariant under different levels of labor input. His function is not the same as (2.14) as the costs of establishing the crop (and therefore acknowledging a labor input) are central to the analysis. It would be safe to assume his yield function corresponds to a fixed-coefficients form[20] while later authors have admitted the possibility of substitution between the labor input and the rotation length.

With all parameters as previously defined, equation (2.19) may be rewritten for an infinite series of rotations:

$$\underset{L,\tau}{\mathrm{MAX}}\left\{pF(L;\tau)e^{-\rho\tau} - \omega L\right\}\left\{1 + e^{-\rho\tau} + e^{-2\rho\tau} + \ldots\right\} \quad \ldots \quad (2.22)$$

which is equivalent to:

$$\underset{L,\tau}{\mathrm{MAX}}\left\{\frac{pF(L;\tau)e^{-\rho\tau} - \omega L}{1 - e^{-\rho\tau}}\right\} \quad \ldots \quad (2.23)$$

The solution to (2.23) is the present value of all future net revenues (discounted). With competitive markets it is also equal to the value of land under forest cultivation. Under management, the land will yield a perpetual series of periodic net revenues, equivalent to a revenue of R per time period and therefore:

$$\frac{R}{\rho} = \underset{L,\tau}{\mathrm{MAX}}\left\{\frac{pF(L;\tau)e^{-\rho\tau} - \omega L}{1 - e^{-\rho\tau}}\right\} \quad . . . (2.24)$$

The maximization (2.23) leads to interesting marginal conditions. Denoting L^*, τ^* as the solution:

$$\left.\frac{p\partial F(L;\tau)}{\partial L}\right|_{L=L^*} e^{-\rho\tau^*} = \omega \quad . . . (2.25)$$

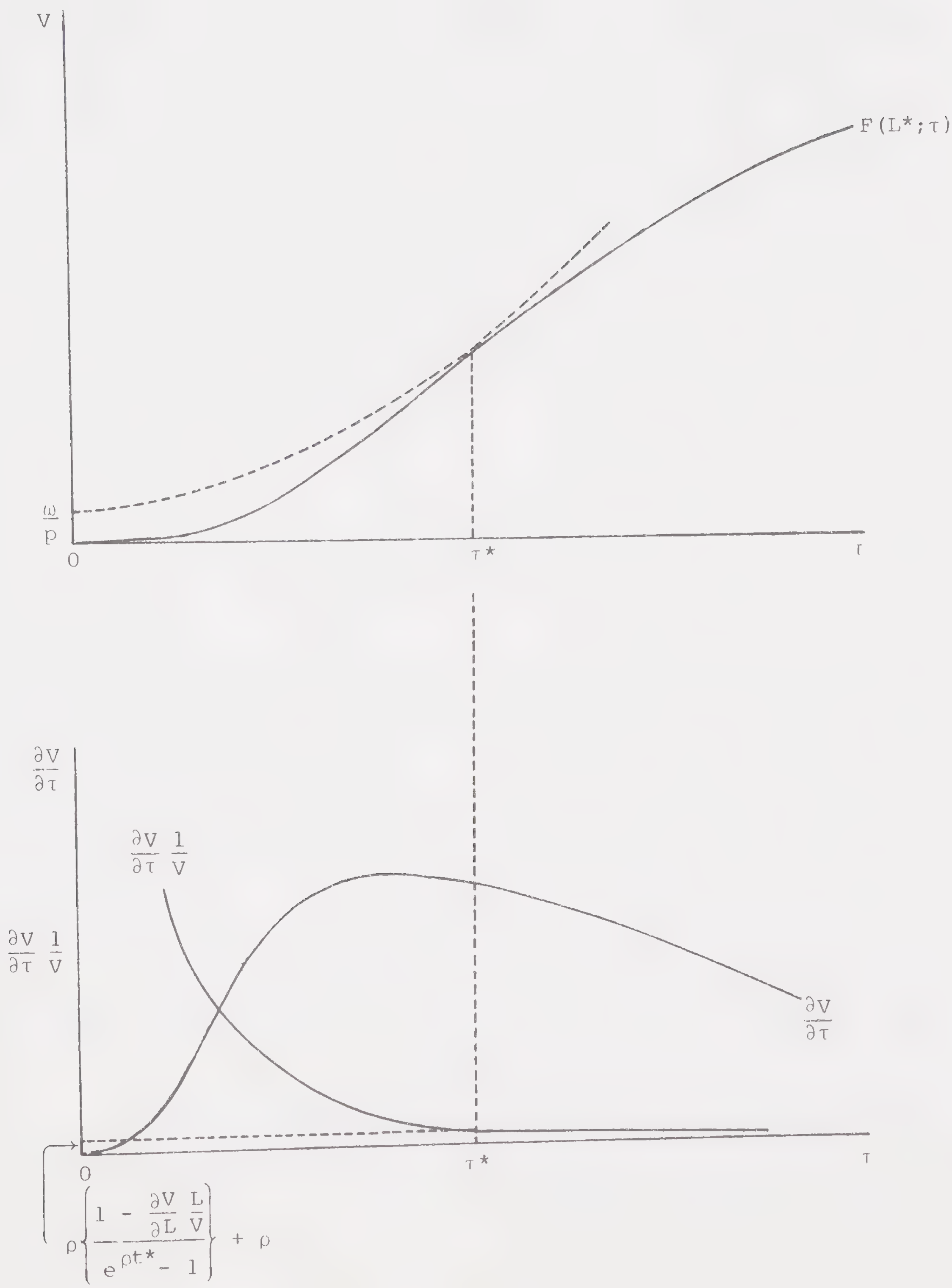
$$p\left.\frac{\partial F(L;\tau)}{\partial \tau}\right|_{\tau=\tau^*} = \rho\left\{\frac{pF(L^*;\tau^*)e^{-\rho\tau^*} - \omega L^*}{1 - e^{-\rho\tau^*}} + pF(L^*;\tau^*)\right\} \quad . . . (2.26)$$

Equation (2.25) is identical to (2.21), namely, that the discounted value of the marginal product must equal the wage rate. Equation (2.26) is easily simplified by noting (2.24) and substituting:

$$p\left.\frac{\partial F(L;\tau)}{\partial \tau}\right|_{\tau=\tau^*} = R + \rho pF(L^*;\tau^*) \quad . . . (2.27)$$

See Figure 2.2.

Figure 2.2



Assuming the entrepreneur does not own the land and therefore pays the rental fee R during each time period, the left hand side is simply the marginal revenue of delaying the harvest one time period while the RHS is the opportunity cost of that delay. Equation (2.26) is therefore simply a marginal revenue, marginal cost solution.

Equation (2.26) may be rewritten by dividing through by $pF(L^*;\tau^*)$ and appealing to (2.25). The notation has been changed for convenience however the equation is still evaluated at $(L^*;\tau^*)$:

$$\frac{\partial V}{\partial \tau}\frac{1}{V} = \rho\left\{\frac{1 - \frac{\partial V}{\partial L}\frac{L}{V}}{e^{\rho\tau} - 1}\right\} + \rho \qquad \ldots \quad (2.28)$$

In this model, if the elasticity of volume with respect to the labor input is unity, the result is the same as the previous model.

This model yields several comparative static results:[21]

(a) $\frac{\partial L}{\partial \rho} < 0, \; \frac{\partial \tau}{\partial \rho} > 0 \qquad \ldots \quad (2.29)$

These results indicate that an increase in the rate of interest will lead to a longer rotation and that

the entrepreneur will reduce the labor used to establish the crop. Substitution between the labor input and the rotation length is implied. It should be noted however, that the rotation length is a parameter and not a factor of production. The term substitution is therefore used in a slightly different context than is understood in economics.

(b) $\frac{\partial L}{\partial \omega} < 0, \frac{\partial \tau}{\partial \omega} > 0$ . . . (2.30)

As most economists would predict, an increase in the factor price will lead to substitution away from that factor. In this analysis, an increase in the wage rate will result in a longer rotation.

(c) $\frac{\partial L}{\partial p} > 0, \frac{\partial L}{\partial p} < 0$ . . . (2.31)

These findings indicate that an increase in the sale price of timber will result in a shorter more intensive rotation. The entrepreneur will utilize more labor and allow the crop to be harvested on a shorter rotation.

Concluding Comments

In the previous sections of this chapter the

technology, assumptions and the results which follow from the assumptions have been detailed. Indeed, the assumptions are the key to the analysis and as such deserve some discussion.

Competitive markets with profit or wealth maximization are assumptions largely traditional to economic analysis. Maximum sustained yield plays a similar role in the history of thought on forestry. The analysis could be undertaken using different assumptions of market structure and behavior.

Perhaps the most heroic assumption in this analysis is that parameters do not change over time. The mathematics also imply that the entrepreneur views the riskless future with certainty. Expectations could be introduced and the model made probabilistic. Mathematical manipulation will then perhaps overshadow the nature of forest investment. The result of this type of approach would be a loss of simplicity.

Although the model lacks realism it is still insightful when one considers the state of the industry today. Commercial forest land is either private or publicly owned. On private land the trend has been toward shorter more intensive rotations. The analysis suggests that this

type of substitution may result from increases in the sale price of timber, *ceteris paribus*. On public lands the entrepreneur may be limited in his choice of rotation lengths. If this constraint binds the entrepreneur to a longer rotation than he would otherwise choose, the model would suggest the rental payments must be lower in order to compensate. In this case, one can easily envision the development of a system with lower rental payments and regulations to assure establishment of the new crop after harvesting. In the face of such controls, the entrepreneur may minimize the cost of establishing the new crop.

Footnotes

[1]A good discussion of four types of production including point-input, point-output can be found in:

Tisdell, C. A., *The Theory of Economic Allocation* (Sydney: Wiley and Sons, 1972), p. 331.

[2]This implies that labor demands payment immediately for services. This appears to be a reasonable assumption.

[3]Farming cereal crops is a point-input, point-output process in which the entrepreneur has little or no control over the length of the period of production.

[4]Wicksell, K., *Lectures on Political Economy*, Vol. 1, translated by L. Classen (London: Routledge and Sons, 1934), p. 181.

[5]*Ibid.*

[6]MacKinney, F. X., *et al*, "Construction of Yield Tables for Non-normal Loblolly Pine Stands," *J. Agric. Res.* 54 (1937), pp. 534-535.

[7]Note that this function violates the first property in the previous quotation. It is, however, simple to estimate.

[8]*Op. cit.*, p. 353.

The logistic growth function:

$$V = K/\left\{1 + e^{\overline{\alpha} - \overline{\beta}\tau}\right\}$$

has the property that the inflection point always occurs at ½K where K is the upper asymptote. For this reason it is rejected for use in forestry.

[9]One possible specification of this function is:

$$V = e^{(\alpha - \beta/\tau)} \cdot A\left\{\delta L^{-z} + (1 - \delta)\right\}^{-\frac{1}{z}}, \; A > 0, \; 0 \le \delta \le 1, \; z \ge -1.$$

Therefore, if $L = 0$ and $A = \dfrac{1}{(1 - \delta)^{-\frac{1}{z}}}$, this reduces to equation (2.15): $V = e^{(\alpha-\beta/\tau)}$. For $L > 0$, $\tau > 0$:

$$\frac{\partial^2 V}{\partial L \partial \tau} = \frac{\partial^2 V}{\partial \tau \partial L} > 0 \text{ and: } \frac{\partial^2 V}{\partial \tau^2} \gtrless 0 \text{ as } \tau \lessgtr \frac{\beta}{2}.$$

[10]See:

Davis, K. P., *Forest Management: Regulation and Valuation*, Second Edition (New York: McGraw-Hill, 1966), p. 226.

[11]Davis, K. P., *Forest Management: Regulation and Valuation* (1966), p. 6.

[12]Metzler, L. A., "The Rate of Interest and the Marginal Product of Capital," *J.P.E.* 58 (1950), p. 293.

[13]Davis, K. P., *Forest Management: Regulation and Valuation* (1966), p. 100.

[14]Böhm-Baverk, Eugen von, *Capital and Interest*, Vol. 3 (1909), translated by Hans Sennholz (Illinois: Libertarian Press, 1958), p. 12 and footnote 32.

[15]*Op. cit.*, p. 178.

[16]Faustman, M., "On the Determination of the Value which Forest Land and Immature Stands Possess for Forestry," (1849), editor English edition M. Gane, *Martin Faustman and The Evaluation of Discounted Cash Flow* (University of Oxford: Commonwealth Forestry Institute, Institute Paper 42, 1968).

[17]Gaffney, M., *Concepts of Financial Maturity of Timber and Other Assets*, Agricultural Economics Information Series 62 (Raleigh, N.C.: North Carolina State College, 1957).

[18]Hirshleifer, J., "On the Theory of The Optimal Investment Decision," *J.P.E.* 66 (1958): 198-209.

[19]Samuelson, P. A., "Economics of Forestry in an Evolving Society," paper delivered at Symposium: "The Economics of Sustained Yield Forestry," University of Washington, Seattle, Washington, 1974.

[20]See:

Nicholson, W., *Microeconomic Theory: Basic Principles and Extensions* (Illinois: Dryden Press, 1972), p. 208.

[21]The comparative static results are from:

Jackson, D., "On the Microeconomics of Timber: A Maturing Asset." Unpublished doctoral dissertation. Seattle: University of Washington, 1975, pp. 23-25.

The growth function upon which these results are derived is concave in τ. The growth function defined in this paper has one inflection point. Therefore, the comparative static results are applicable here only if the solution to (2.23) occurs at $\tau^* > \tau'$ where τ' is the inflection point, since the volume function is concave in τ for those points.

The sufficiency conditions for an interior maximum are:

$$\frac{\partial^2 V}{\partial L^2} < 0$$

and:

$$\frac{\partial^2 V}{\partial L^2}\left\{\frac{\partial^2 V}{\partial \tau^2} - \rho\frac{\partial V}{\partial \tau}\right\} > \left\{\frac{\partial^2 V}{\partial L \partial t} - \rho\frac{\partial V}{\partial L}\right\}^2$$

Since the right hand side of this inequality is always positive and:

$$\frac{\partial^2 V}{\partial L^2} < 0 \text{ for all } L,$$

if the sufficiency inequality holds for some L^*, τ^*, it must be true that:

$$\frac{\partial^2 V}{\partial \tau^2} - \rho\frac{\partial V}{\partial \tau} < 0$$

if: $V = e^{(\alpha-\beta/\tau)} A\left\{\delta L^{-z} + (1 - \delta)\right\}^{-\frac{1}{z}}$, then it must be true at L^*, τ^* that:

$$\frac{\beta - 2\tau^*}{(\tau^*)^2} < \rho$$

Let us assume that $\tau^* = \tau' = \beta/2$. This requirement becomes $\rho > 0$. A maximum at L^*, $\tau^* = \tau'$ is therefore very possible. If a maximum occurs at $\tau' - \varepsilon$ then:

$$\frac{8\varepsilon}{(\beta - 2\varepsilon)^2} < \rho$$

if $\varepsilon = 1$ and $\beta = 10$, this requires that $\rho > .125$. A low β is therefore assumed. This assumption simply means that growth is slow which is not unreasonable for many areas. Jackson's comparative static results hold unless the maximum occurs at the inflection point. This point is assumed away in order to present the results.

The reader may ask why the author has invited this problem by defining this type of growth function. The inflection point is theoretically correct (albeit insignificant), and required for the proper development of the dynamic models of the next chapter.

CHAPTER THREE

DYNAMIC MODELS OF RESOURCE EXPLOITATION

The purpose of this chapter is to review three dynamic models of renewable resource exploitation. Although the models are directed toward the common property resource, they are still of interest to this study.

Plourde's model[1] is perhaps the simplest of the three but at the same time on the basis of the conclusions that may be drawn, it is the most powerful. The resource planner maximizes welfare which is given by the sum of all intertemporal discounted utilities. Under this framework the nature of the steady-state that will evolve is easily examined and the state can be rather explicitly determined. The problem is formulated as a current-valued Hamiltonian[2] and solved using the Maximum Principle of Pontryagin *et al.*[3]

Smith's model[4] of the appropriated forest resource is static in nature. This model assumes the entrepreneur maximizes profits and investigates the nature of the static equilibrium. A dynamic model based on similar

assumptions will then be examined and the results of the two models contrasted. One of the most important features of both models is the cost function. In the static model, this function plays a key role in determining the equilibrium while in the dynamic model the discount rate may be more important in determining the steady state.

In Brown's model,[5] production is determined by a production function incorporating a variable factor and the stock of the resource. The cost of the production is determined by the factor payment rate and the quantity of the variable factor employed. The planner maximizes discounted net revenues for all time or, alternatively, wealth. The problem is again formulated as a current-valued Hamiltonian.

All three models assume that the growth function of the resource is given by the logistic function. The nature of the function and its stability properties will be considered first.

The Logistic Function

The logistic function or "Law of Population Growth"[6] may be written:

$$N = \frac{k}{1 + e^{\alpha - rx}} \qquad \ldots (3.10)$$

where x denotes time from the population's origin or the age of the population. N therefore is the number of individuals comprising the population.

The function is often used to approximate the mass growth of an individual.[7] It has also been used to approximate the mass growth of a population.[8] In the following exploitation models the latter assumption is made. N is in these cases the mass of the resource while x is again the age of the resource or time since its origination.

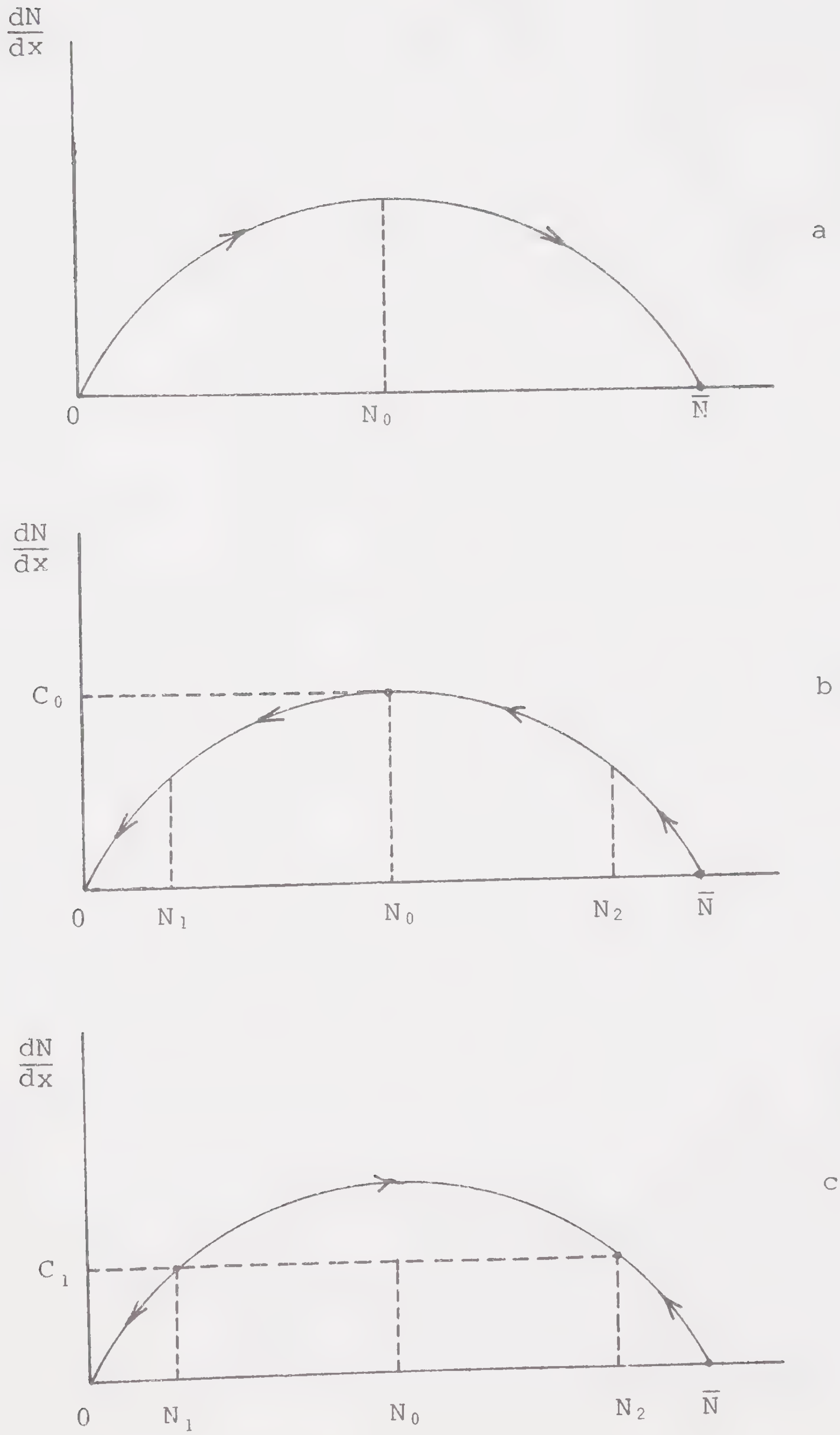
The function (3.10) is an "S" shaped or sigmoid function with an upper asymptote of k. The inflection point always occurs at $\frac{k}{2}$; when $x = \frac{\alpha}{r}$. The curve is symmetric about the inflection point.

It should be noted that the logistic function is not in itself characteristic of growth but that the sigmoid form is the essential feature. The logistic function, equation (2.15) and the so-called "Gompertz Curve"[9] are just examples of the family of sigmoid functions. The logistic curve does have desirable characteristics, one of which is the fact that the first derivative is easily and simply expressed as a function of the mass itself:[10]

$$\frac{dN}{dx} = rN - \frac{r}{k}N^2 \qquad . \; . \; . \; (3.11)$$

The stability properties of (3.11) should be considered. In view of (3.10) it is evident that $N > 0$ for $x = 0$ and therefore the complete absence of the resource will not biologically exhaust the resource. One stable equilibrium does exist at $N = k$ and for any mass of the resource different from k, the resource mass will grow or decline, approaching k. These properties are graphically displayed in Fig. 3.1a. Further properties of the sigmoid growth function will be illustrated using Fig. 3.1b. Assume a policy is instituted which allows for the harvest of C_0 units of the resource in each instant of time. Further, assume that initially the stock of the resource is N_2. Since $C_0 > (dN/dx)|N = N_2$, the stock of the resource will fall until $C_0 = (dN/dx)|N = N_0$ and the stock will remain at N_0. Similarly, with the initial stock N_1, $C_0 > (dN/dx)|N = N_1$ and over time the entire stock of the resource will be harvested. The equilibrium at $C_0 = (dN/dx)|N = N_0$ is not stable however. If the stock should ever fall below N_0, the entire stock will be harvested over some period of time. If the stock momentarily shifts above N_0, equilibrium will be restored.

Figure 3.1



$\frac{dN}{dx}$
0
N_0
$\bar{N}$
a
$\frac{dN}{dx}$
C_0
0
N_1
N_0
N_2
$\bar{N}$
b
$\frac{dN}{dx}$
C_1
0
N_1
N_0
N_2
$\bar{N}$
c

Suppose a policy is instituted which calls for the harvest of C_1 units of the resource in each instant of time. Fig. 3.1c illustrates the movement of the system. If the resource stock is initially at $N > N_1$ and with a harvest of C_1 in each instant of time, the resource stock will equilibrate at $N = N_2$. This equilibrium is stable for all $N > N_1$ since $C_1 < (dN/dx)|N_1 < N < N_2$ and $C_1 > (dN/dx)|N > N_2$. Another equilibrium exists at $C_1 = (dN/dx)|N = N_1$. This equilibrium is not stable however since a momentary change in the resource stock will result in complete harvest of the resource stock or the establishment of the stable equilibrium at $N = N_2$. The former case arises when $N < N_1$ since $C_1 > (dN/dx)|N < N_1$ while the latter occurs when $N > N_1$ because $C_1 < (dN/dx)|N_1 < N < N_2$.

Diagrams of this type are not novel to dynamic resource models, the reader familiar with Neoclassical growth models will recognize them.[11] Indeed, they may be interpreted in another manner altogether and some insight gained into the nature of forest policy. It has been shown that with any exploitation at all, only one stable equilibrium exists.[12] However, this is only a result of the assumed policy setting the harvest at some level for all time. Recall the derivation of (3.11) in footnote 10. For each N or resource stock given in (3.11) there corresponds a unique x or age. Suppose a policy

was instituted (see Fig. 3.1c again) that set the resource stock at N_1 for all time. Associated with N_1 is an age, call it x_1. The policy could work this way.[13] The authority would simply decree that all individuals of the resouce may be harvested at age x_1 but at no earlier age. In the case of forestry, this would simply mean that no acre may be harvested until age x_1. Similarly, for fisheries, setting the mesh size of nets and noting that age and size will be highly correlated, the result will be the same. Consider the stabilizing effect of such a policy on the system. Assume that initially the stock of the resource is $N > N_1$. The policy is instituted and all individuals of age $x > x_1$ are harvested. Equilibrium is established at N_1 with $C_1 = (dN/dx)|N = N_1$.[14] Suppose the stock falls below N_1 for an instant, the policy becomes very effective, C may fall for a period of time but the equilibrium will be restored. The equilibrium is stable. Notice that it is the policy which determined the stability of the system.

Plourde's Simple Model

The problem is simply stated:

> Suppose a planner wishes to control the consumption of natural resources so as to maximize the welfare functional:

$$\int_0^{\infty} U[C(t)]e^{-\delta t}dt.^{15} \qquad \ldots (3.12)$$

The cardinal utility function has as its argument C(t) or the harvest (catch, cut) from the resource. It is implicitly assumed that the utility at any instant of time can be added to utilities from other instants of time to yield a measure of welfare. The positive discount rate signifies decreasing rates of substitution of future consumption for present consumption as the future recedes.

Plourde's Hamiltonian is therefore:

$$H[C(t),N(t),p(t),t] = \left\{U[C(t)] + p(t)[rN(t) - \frac{r}{k}N(t)^2 - C(t)]\right\}e^{-\rho t} \quad \ldots (3.13)$$

the solution of which is equivalent to the solution of the current-valued Hamiltonian:

$$H[C(t),N(t),p(t)] = U[C(t)] + p(t)[rN(t) - \frac{r}{k}N(t)^2 - C(t)] \quad \ldots (3.14)$$

The properties which the utility function U[C(t)] possess are important. To be consistent with economic theory:

$$U'[C(t)] > 0 \text{ for } C(t) > 0$$

but in addition to the above it is assumed that

$$\lim_{C(t)\to 0} U'[C(t)] = \infty \quad . . . (3.15)$$

This assumption is necessary to assure that the harvest in any moment of time is never zero. An interior solution to (3.14) is therefore guaranteed and the harvest of the resource will be positive in all time periods. Though not stated, the assumption of diminishing marginal utility in this model is required.

Problem (3.14) is constrained by:

$$\frac{dN(t)}{dt} = rN(t) - \frac{r}{k}N(t)^2 - C(t)$$

$$N(t) \geq 0, \; C(t) \geq 0$$

Dropping the time subscript, the necessary conditions for a maximum are[16]:

$$\frac{\partial H}{\partial C} = U'(C) - p = 0 \quad . . . (3.16)$$

and

$$\dot{p} = \rho p - \frac{\partial H}{\partial N} = \rho p - p\left\{r - \frac{2r}{k}N\right\} = p\left\{\rho - r + \frac{2r}{k}N\right\}. . . (3.17)$$

In addition, the transversality conditions must also be

satisfied:

$$\lim_{t \to \infty} e^{-\rho t} pN = 0, \quad \lim_{t \to \infty} e^{-\rho t} p = 0 \qquad \ldots (3.18)$$

The transversality conditions require that the present value of the resource stock ($e^{-\rho t}pN$) and the present value of a marginal unit of resource stock ($e^{-\rho t}p$), in the limit, equal zero.

Note that from (3.15), C may be expressed as a function of p:

$$C = C(p) \qquad \ldots (3.19)$$

On the basis of the utility function assumptions, (3.19) is a single-valued function.

The nature of the steady-state solution can be investigated by examining the system of differential equations:

$$\frac{dN}{dt} = rN - \frac{r}{k}N^2 - C(p) \qquad \ldots (3.20)$$

$$\frac{dp}{dt} = p\left\{\rho - r + \frac{2r}{k}N\right\}$$

First, when $dp/dt = 0$, either $p = 0$ or:

$$\rho - r + \frac{2r}{k}N = 0 \qquad . . . (3.21)$$

In view of (3.15), $p = 0$ is not possible since by (3.16) this would mean $U'(C) = 0$, a contradiction since $U'(C) > 0$ by assumption. Therefore, $p > 0$ when $dp/dt = 0$. Denoting the steady-state values of N,p,C as N^*, p^*, C^*; N^* is given by (3.21).

Secondly, from (3.20), when $dN/dt = 0$:

$$rN - \frac{r}{k}N^2 - C(p) = 0$$

Along this curve:

$$\frac{dC}{dp}\frac{dp}{dN} = r - \frac{2rN}{k}$$

and therefore:

$$\left.\frac{dp}{dN}\right|_{\dot{N}=0} = \frac{r - 2rN/k}{C'(p)} \qquad . . . (3.22)$$

where $\dot{N} = dN/dt$.

From (3.16) and recalling that positive but diminishing marginal utility is assumed, $C(p)$ is a decreasing function of p and therefore $C'(p)$ is negative. For

$N < k/2$, (3.22) is negative.

At this point it is very instructive to pause and consider the results of the model. It is easily seen that $k/2$ is the inflection point in the logistic function. We also know that the equilibrium level of the resource stock is given by (3.21) and therefore:

$$N^* = \left\{\frac{r - \rho}{r}\right\}\frac{k}{2} \stackrel{<}{\underset{>}{=}} \frac{k}{2} \text{ as } \rho \stackrel{>}{\underset{<}{=}} 0 \qquad . . . (3.23)$$

Assuming that $\rho > 0$ then the steady-state level of the resource stock is at a level less than that given by the inflection point on the function. Figure 3.2a illustrates this point.

From (3.20), for fixed N, and recalling C(p) to be a decreasing function of p, it follows that

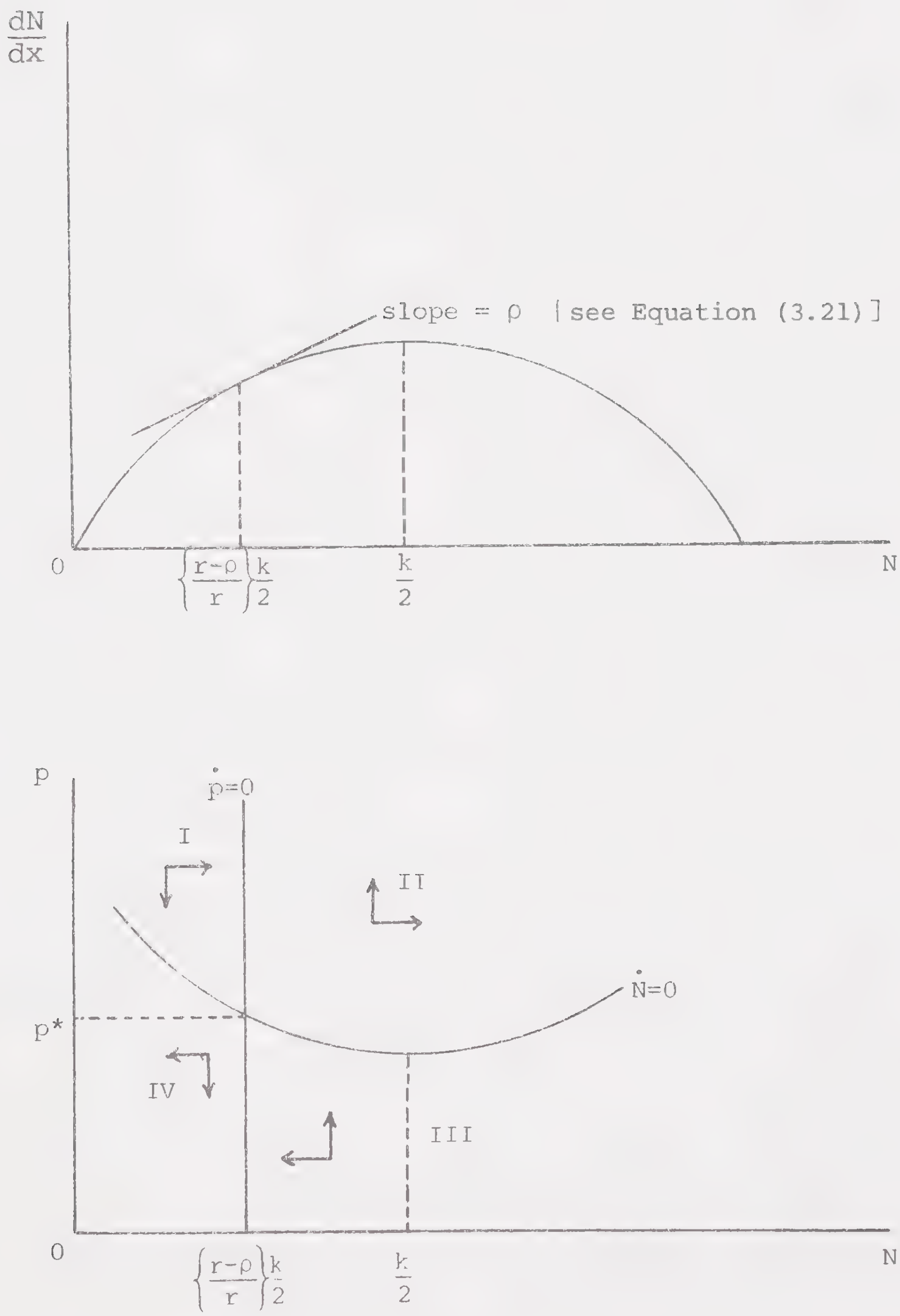
$$rN - \frac{r}{k}N^2 - C(p) = 0$$

is an increasing function of p. Therefore below the curve $dN/dt = 0$, $dN/dt < 0$ and above it $dN/dt > 0$. Similarly for (3.17) for fixed p:

$$p\left\{\rho - r + \frac{2r}{k}N\right\} = 0$$

is a decreasing function of N. Therefore:

Figure 3.2


dN/dx
slope = ρ [see Equation (3.21)]
0
$\left\{\frac{r-\rho}{r}\right\}\frac{k}{2}$
$\frac{k}{2}$
N
p
$\dot{p}=0$
I
II
$\dot{N}=0$
p*
IV
III
0
$\left\{\frac{r-\rho}{r}\right\}\frac{k}{2}$
$\frac{k}{2}$
N

a

b

$$\frac{dp}{dt} \gtrless 0 \text{ as } N \lessgtr \left\{\frac{r - \rho}{r}\right\}\frac{k}{2}$$

Denoting $dp/dt > 0$ with upward arrows and $dN/dt > 0$ with arrows pointing to the right, the motion of all points in phase space is indicated in Figure 3.2b.

Consider the points in region II. Suppose initially, $N(0)$, $p(0)$ are in this region. Then, since $\dot{N} > 0$, $\dot{p} > 0$ equilibrium at N^*,p^* is not possible. Any path entering region II at any time will never reach N^*,p^*. Similarly, with initial $N(0)$, $p(0)$ in region IV we note that $\dot{N} < 0$, $\dot{p} < 0$. The resource is wiped out. Any path entering region IV will never achieve the steady-state N^*,p^*. Any point initially in region III or I may be on the path eventually reaching equilibrium. Any path in region I or III which never leaves these regions is an optimal path for if a path enters region II or IV it will never reach N^*,p^*.

Noting that the Lagrangian multiplier p may be interpreted as the price of a marginal unit of the resource stock, the two major conclusions of this model are:

(1) with a positive discount rate, the optimal steady-state level of the resource stock is less than that given by the inflection point on the growth curve and,

(2) the planner can assign the initial price $p(0)$ which will guide the exploitation of the resource optimally.

Therefore, when the stock of the resource is high and a low price imputed to that stock, exploitation will reduce the stock. When the stock is low, a high imputed price will assure conservation. Given the initial stock of the resource, the planner should set p so that the initial point is in region I or III of the phase diagram.

Finally, if $\rho < r - \frac{2r}{k}N$ the resource will be exhausted.

Smith's Model of Exploitation

Smith's article[17] contains a comprehensive study of dynamic models of resource exploitation. Both renewable and non-renewable resources are considered. The major thrust of the paper is in the area of resource externalities, specifically crowding and stocking externalities. Crowding externalities occur as a result of the common property nature of the resource. As the number of individuals (or firms) exploiting the resource increases, the costs of exploitation rise. Smith also assumes that stocking externalities influence the cost function of the firm. As the stock of the resource increases, capture is assumed easier and costs fall.

The article contains a model of the unappropriated forest resource. This assumption is unrealistic since tenure systems exist in forestry. A static model of the

appropriated forest resource is also contained within the paper. The first task of this section will be to review this model. A dynamic model based on similar assumptions will then be introduced.

Smith assumes that crowding and stocking externalities do not exist in forest exploitation. The latter assumption, though not critical, is questionable since one would expect costs to fall as the volume density of the forest rises. In the interest of comparison of the results, the dynamic model presented will retain his assumption. The number of units of physical capital (for example, boats or harvesting machines) is denoted as (K). Each unit of capital produces output at the rate (X) and the price of such an output unit (P) is assumed constant. The resource grows according to the growth function (3.11) which for convenience will be written:

$$\frac{dN}{dx} = f(N) \qquad \text{. . . (3.24)}$$

The problem the entrepreneur faces is to maximize profits given the cost function:[18]

$$TC = K\phi(X), \quad \phi'(X) > 0, \quad \phi''(X) < 0 \qquad \text{. . . (3.25)}$$

subject to the constraint:

$$\frac{dN}{dt} = f(N) - KX = 0 \qquad \ldots (3.26)$$

It should be pointed out that (3.26) is binding and the model therefore static. In addition, the entrepreneur decides how many units of capital and at what output rate they will operate.

Smith's Lagrangian is therefore:

$$\psi = PKX - K\phi(X) + \gamma\left\{f(N) - KX\right\} \qquad \ldots (3.27)$$

The necessary conditions for this problem are:

$$\frac{\partial\psi}{\partial K} = PX - \phi(X) - \gamma X = 0 \qquad \ldots (3.28)$$

$$\frac{\partial\psi}{\partial N} = \gamma f'(N) = 0 \qquad \ldots (3.29)$$

$$\frac{\partial\psi}{\partial X} = P - \phi'(X) = \gamma \qquad \ldots (3.30)$$

$$\frac{\partial\psi}{\partial\gamma} = f(N) - KX = 0 \qquad \ldots (3.31)$$

The major point of this model involves (3.29). If the constraint is effective and $\gamma \neq 0$, then $f'(N) = 0$. The entrepreneur will therefore set the harvest rate equal

to the highest growth rate possible. Referring back to Fig. 3.1b, the stock of the resource in equilibrium will be N_0 with harvest rate C_0. This implies that the forest would be managed to maximize sustained yield. Substituting (3.30) into (3.28), eliminating γ, a further necessary condition is:

$$\frac{\phi(X)}{X} = \phi'(X) \qquad \ldots (3.32)$$

or average and marginal costs are equal. The firm will therefore operate each unit of physical capital at minimum average cost. Looking at (3.30):

$$P - \gamma = \phi'(X)$$

the reader will note that $\gamma > 0$ and $P > \phi'(X)$ since only this case yields non-negative profits. In fact, since $P > \phi'(X)$, this model suggests that the firm will be earning profits in equilibrium. The Lagrangian multiplier may be interpreted as the marginal profit associated with an increase in the resource stock.

One further point should be made concerning this model. Equation (3.32) determines the intensive margin for the firm; that is, each boat or machine will be operated at minimum average cost. The constraint (3.36) and

equation (3.29) determine the extensive margin for the firm or the rate of output.

The basic assumptions of this model can easily be considered in a dynamic framework. For this purpose it will be convenient to consider the firm's control variable as the rate harvest of the resource in each time period. Therefore, denoting the catch or cut as ($C \equiv KX$) and assuming the entrepreneur maximizes discounted profits subject to the constraint:

$$\frac{dN}{dt} = f(N) - C \qquad \text{. . . (3.33)}$$

the appropriate current-valued Hamiltonian is:

$$H(N,C,\lambda) = PC - h(C) + \lambda\left\{f(N) - C\right\} \qquad \text{. . . (3.34)}$$

where the cost function $h(C)$ is given by $h(C) \equiv K\phi(X)$. In addition to (3.33), the problem is constrained by:

$$N \leq 0, \; C \geq 0$$

The necessary conditions for a maximum are:

$$\frac{\partial H}{\partial C} = P - h'(C) - \lambda = 0 \qquad \text{. . . (3.35)}$$

$$\dot{\lambda} = \lambda\left\{\rho - f'(N)\right\} \qquad \text{. . . (3.36)}$$

where ρ is the discount rate. It is also required that the transversality conditions be satisfied:

$$\lim_{t \to \infty} \lambda e^{-\rho t} = 0, \quad \lim_{t \to \infty} \lambda N e^{-\rho t} = 0$$

Equation (3.36) can be solved for the steady state level of the resource stock by setting $\dot{\lambda} = 0$. Therefore, either $\lambda = 0$ or $\rho = f'(N)$. It is not possible to qualitatively determine which condition holds in the steady state. Notice however, that if $\lambda \neq 0$ then $\rho = f'(N)$ which is the same result the Plourde model yields. If $\lambda = 0$ then the constraint (3.33) is not binding and (3.35) indicates that price and marginal cost must be equated.

The point of introducing Smith's static model and then a dynamic model was simply to bring to bear the influence of time on the solution. The static solution indicates maximum sustained yield while the dynamic approach confirms Plourde's result that the rate of growth and the interest rate determine the optimal steady state resource stock.

Brown's Model[18]

To this point, a dynamic model based on utility and one centered on a cost function have been examined. The last model considers the same exploitation problem

but from the point of view of a production function.

Production from the resource is given by the production function:

$$C = G(N,K) \qquad \text{. . . (3.37)}$$

which is assumed to be homogeneous of the first degree with C as the catch or cut and N the stock of the resource. K may be considered capital or boats as in the previous model. The output from (3.37) is valued at the constant rate P and the variable factor K is priced at the constant factor price ω.

As with the previous models, the resource grows according to the logistic function (3.10) in the absence of exploitation. For the purposes at hand the resource will be assumed to be appropriated. The planner therefore is assumed to maximize discounted profits:

$$\int_0^\infty \left\{ PG(N,K) - \omega K \right\} e^{-\rho t} dt \qquad \text{. . . (3.38)}$$

In view of the homogeneity assumption the problem can be rewritten:

$$\int_0^\infty \left\{ Pg(k) - \omega k \right\} N e^{-\rho t} dt \qquad \text{. . . (3.39)}$$

where:

$$g(k) = G\left(1, \frac{K}{N}\right)N, \; k \equiv \frac{K}{N}$$

The current-valued Hamiltonian is therefore:

$$H = \left\{\left(Pg(k) - \omega k\right)N + \lambda\left(f(N) - Ng(k)\right)\right\} \quad . \; . \; (3.40)$$

The constraints of the problem are:

$$\dot{N} = f(N) - Ng(k) \qquad . \; . \; . \; (3.41)$$

$$C \geq 0, \; N \geq 0$$

The necessary conditions for a maximum are:

$$\frac{\partial H}{\partial k} = (P - \lambda)g'(k) - \omega = 0 \qquad . \; . \; . \; (3.42)$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial N}$$

$$= \rho\lambda - \left\{(P-\lambda)[g(k) - kg'(k)] + \lambda f'(N)\right\} \quad . \; . \; . \; (3.43)$$

and the transversality conditions:

$$\lim_{t \to \infty} N\lambda e^{-\rho t} = 0, \; \lim_{t \to \infty} \lambda e^{-\rho t} = 0$$

Looking at equation (3.42) and interpreting λ as the imputed value of a unit of unharvested resource, $(P - \lambda)$ can be viewed as a net price. Suppose the planner imputes no value to an additional unit of the resource stock. Equation (3.42) simply means the value of the marginal product of the variable factor must equal its factor price. If a high value of λ is imputed the value of the marginal product falls and therefore the rate of exploitation or harvest should fall. The interpretation of λ allows clear insight into the problem of common property. Each user of the resource does not value the stock of the resource since there is no guarantee that a unit of the resource stock left unharvested will be available in any future time period. The value of λ is therefore zero and the resource is depleted at a faster rate than optimal.

Equation (3.43) can be rewritten:

$$\frac{\dot{\lambda}}{\lambda} + f'(N) = \rho - \left\{\frac{P}{\lambda} - 1\right\}\frac{\partial G}{\partial N} \qquad . . . (3.44)$$

Notice that in the steady-state with $\dot{\lambda} = 0$ if the imputed value of a unit of the resource stock equals the market price then (3.44) requires $f'(N) = \rho$. A general interpretation of (3.43) can be given if written:

$$\dot{\lambda} + \frac{\partial H}{\partial N} = \rho\lambda$$

In this form,[19] the interpretation is that the sum of the marginal productivity of the resource ($\partial H/\partial N$) and capital gains ($\dot{\lambda}$) must equal the interest on the investment of an additional unit of resource stock ($\rho\lambda$).

The phase diagram analysis for this model will not illuminate any new points. It will suffice to state that $(d\lambda/dN)|\dot{N} = 0 > 0$ and $(d\lambda/dN)|\dot{\lambda} = 0 < 0$ for this problem.

Sufficiency Conditions

The sufficiency conditions for the three dynamic models are satisfied if[20]:

$$H^0(N,\lambda,t) = \max_{C} H\ (N,C,\lambda,t)$$

is a concave function of N given N,λ,t where λ is the co-state variable.

In both Plourde's model and the dynamic model using Smith's assumptions H^0 is concave for all N, while in Brown's model the condition is satisfied if:

$$P \geq \lambda.$$

Concluding Remarks

The three dynamic models reviewed in this chapter all utilize different approaches and are somewhat similar to static utility, cost and production analysis. Plourde's model gives rather definite results. The steady state level of the resource stock is unambiguously determined. The adapted dynamic model of Smith's static analysis yields the same steady state resource stock as the Plourde model but the assumption that the co-state variable is not equal to zero is required. Smith's static model indicates that the resource should be managed to maximize sustained yield. The effect of introducing time and a discount rate into the analysis changes this prescription. In both models in the Smith section, the cost function had output or harvest as its argument. Stock externalities were not present. Brown's model introduces this assumption by including the stock of the resource as an argument in the production function. The results are more difficult to interpret and in general, the steady state resource stock level may be the same as the Plourde model if further assumptions are made.

In all these models, the logistic curve has been used as the basic growth function. In fact, any sigmoid growth function may be used and the same results obtained.

Footnotes

[1]Plourde, C. G. "A Simple Model of Replenishable Resource Exploitation," *A.E.R.* 50 (1960): 518-522.

[2]Arrow, K., "Applications of Control Theory to Economic Growth," *Stanford University Inst. Math. Studies Soc. Sci.* (Stanford, California: Stanford University, 1968.)

[3]Pontryagin, L. S. *et al.* *The Mathematical Theory of Optimal Processes* (New York: Interscience, 1962).

[4]Smith, V. "Economics of Production from Natural Resources," *A.E.R.* 58 (1968): 409-431.

[5]Brown, G. "An Optimal Program for Managing Common Property Resources with Congestion Externalities," *J.P.E.* 82 (1974): 163-173.

[6]Lotka, A. J., *Elements of Mathematical Biology* (New York: Dover, 1956).

[7]*Ibid.*, p. 73.

[8]See:
Odum, E. P., *Fundamentals of Ecology*, Second Edition (London: Sanders, 1959), 232. k, α, r are positive estimated parameters.

[9]Windson, C. P., "The Gompertz Curve as a Growth Curve," *Proceedings of the National Academy of Sciences*, Vol. 18, no. 1 (1932).

The Gompertz curve is:

$$N = k \exp\left\{-e^{\alpha - \beta x}\right\}$$

[10]From (3.10)

$$\frac{dN}{dx} = \frac{-k(-r)e^{\alpha - rx}}{\{1 + e^{\alpha - rx}\}^2}$$

$$= rN \cdot \frac{e^{\alpha - rx}}{1 + e^{\alpha - rx}}$$

$$= rN \cdot \left\{\frac{k/N - 1}{k/N}\right\} = rN - \frac{r}{k}N^2$$

(Cont'd⟶)

[10]Cont'd:

since $e^{\alpha - rx} = k/N - 1$

[11]See:

Intrilligator, M. D., *Mathematical Optimization and Economic Theory* (Englewood Cliffs, N.J.: Prentice-Hall, 1971), 404.

[12]Recall that for $x=0$, $N>0$.

[13]All individuals in the population must have a lifespan greater than x_1 for the policy to work.

[14]We will retain the assumption implicit in the dynamic models, namely that the policy does not change the production or growth function. This assumption will be examined in Chapter Four.

[15]Plourde, C. G., "A Simple Model of Replenishable Natural Resource Exploitation," 519. The discount rate in this article is δ but to be consistent the discount rate in what follows will be ρ.

[16]For a current-valued Hamiltonian with one state (N) and one control (C) variable the necessary conditions are:

$$\frac{\partial H}{\partial C} - 0, \quad \dot{P} = \rho P - \frac{\partial H}{\partial N}$$

See: Arrow, K., "Applications of Control Theory to Economic Growth," p. 95.

[17]Smith, V., "Economics of Production from Natural Resources."

[18]With this definition of the cost function, which is Smith's, marginal and average costs are never equal. The necessary conditions which follow cannot therefore be satisfied.

[19]Brown, G., "An Optimal Program for Managing Common Property Resources with Congestion Externalities."

[20]See:

Arrow, K., "Applications of Control Theory to Economic Growth," p. 95.

[21]*Ibid.*

CHAPTER FOUR

A PROBLEM AND CONCLUDING COMMENTS

The purpose of this chapter is three-fold. Firstly, this chapter will bring together the various conclusions that can be drawn from the dynamic and static models of the previous two chapters. The static models were constructed around a growth or volume function which embodied the characteristics of forest growth. The function was sigmoid and a possible example was given in the second chapter. The dynamic models were based on the logistic growth function. In order to compare the models it is imperative that the same function be used. Therefore, the dynamic models must be reconstructed around the same growth assumptions as the static models. It is easily shown that this alteration does not change the dynamic models in substance. The static models allowed for the possible substitution between the labor input and the rotation length. The labor input was variable and determined as part of the problem. The dynamic models implicitly assume that the labor input is fixed at some level and

therefore labor does not appear in the problem. This point must be reconciled in order to allow comparison.

Secondly, this chapter will bring to light a problem which affects the dynamic models as they pertain to forestry. The problem is essentially one of aggregation. Naturally it would have been desirable to present the problem in an earlier chapter and show the solution to it in succeeding pages. The problem comes to light as more realism is introduced into the models. It is hoped that the discussion of the problem will be lucid enough to allow the reader to judge the relevance of the results from the dynamic models.

Thirdly, this chapter will present my conclusions based on this study.

Steady States in Seven Models

To this point, seven models have been presented. In this section the models will very briefly be reviewed. The static models determine when an acre of forest should be cut, or how long should growth be allowed to continue. These models give, as a result, a rotation age. The dynamic models also answer this basic question. The steady state that results under the assumptions of these models indicates a level of resource stock which would be maintained. At this level of stock in the steady state, the

resultant growth in each time period is cut. Associated with this level of resource stock is an age. In the steady state, this age indicates the oldest acre that will be found in the forest. Therefore, in some respects, the two types of models can be compared. Before such a comparison is undertaken however, certain points must be reconciled.

In general, dynamic models of resource exploitation assume the growth function of the resource is given by the logistic function. The popularity of this form is most likely due to its simplicity.[1] Chapter Three was developed using the logistic function—explicitly in the Plourde model:

$$\frac{dN}{dx} = rN - \frac{r}{k}N^2 \qquad \ldots (3.11)$$

and implicitly in the remaining models of the chapter:

$$\frac{dN}{dx} = f(N) \qquad \ldots (3.24)$$

The transition does not change the models in any substantial manner; the important result of the Plourde model may be given as:

$$\rho = f'(N) \qquad \ldots (4.10)$$

instead of:

$$\rho = N - \frac{2r}{k}N \qquad \ldots (3.21)$$

In fact, any sigmoid function may be used in equation (3.24) without loss of generality. In terms of volume (V) rather than the resource mass (N), it may be written:

$$\frac{dV}{d\tau} = Q(V) = \frac{(\alpha - \log_e V)^2}{\beta} \qquad \ldots (4.11)$$

which follows from:

$$\log_e V = \alpha - \beta/\tau \qquad \ldots (2.15)$$

The reader's attention is brought to the fact that V is defined as volume per acre and in making this transition, the results are now on an acre basis rather than upon the ambiguous definition of N as the resource mass.

Before the static and dynamic results can be compared, one further point must be reconciled. The static models explicitly incorporated a labor input in the volume function:

$$V = F(L;\tau) \qquad \ldots (2.13)$$

However, labor is not considered explicitly in (4.11), (3.24) or (3.11). If the maximization problem given by (2.23) results in the border solution $L^* = 0$, then all results are comparable. $L^* = 0$ simply means that the discounted value of the marginal product of labor is always less than the wage rate. This may not be too far away from the real case in forestry as evidenced by the regulations requiring reforestation. If an interior solution to (2.23) results, the dynamic models and this comparative static model are still comparable if (2.13) for fixed $L = L^*$ is written:

$$V = BG(\tau) \qquad \ldots (4.12)$$

and (4.11) therefore becomes:

$$\frac{dV}{d\tau} = \frac{[\alpha - \log_e(V/B)]^2 \cdot V}{\beta} \qquad \ldots (4.13)$$

In either case, the results are comparable. The author feels that the former case is a more adequate representation of forestry in Canada.

In the north-east quadrant of Figure 4.1, $F(L^*,\tau)$ is graphed. The reader is left to decide whether an interior maximum will result from (2.23).

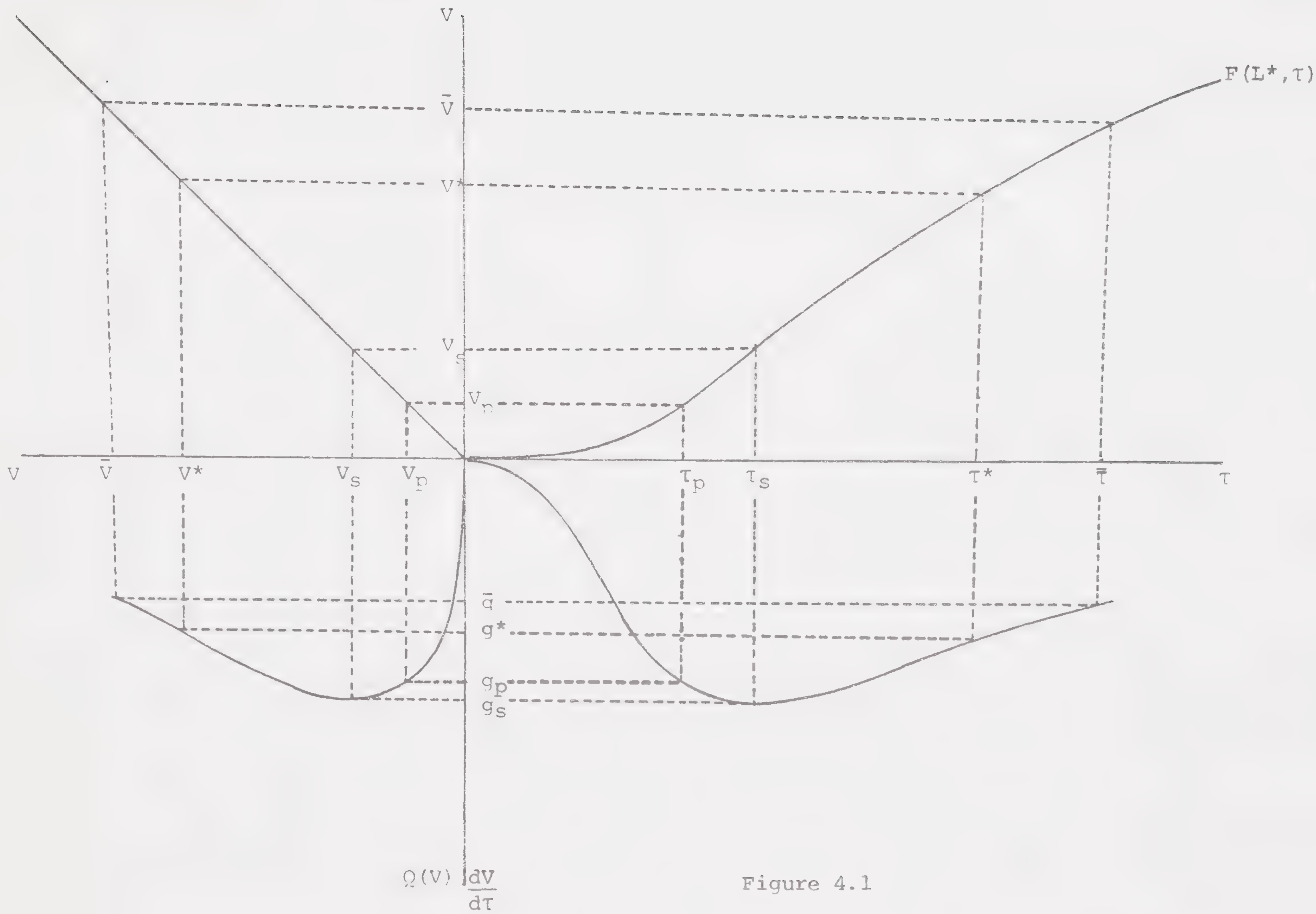


Figure 4.1

The objective of maximizing sustained yield was discussed early in Chapter Two. The problem was given by

$$\underset{\tau}{\mathrm{MAX}}\left\{\frac{F(L^*;\tau)}{\tau}\right\} \qquad . . . (2.16)$$

and the solution by $\bar{\tau}$. On Figure 4.1 the solution is also given by $\bar{\tau}$. The goal of maximizing sustained yield will result in the harvest of $\bar{V}$ units of wood at intervals of $\bar{\tau}$ years on each acre. If a forest contains $\bar{\tau}$ acres, then one acre would be harvested every year. Looking at the south-west quadrant, with a resource stock of $\bar{V}$ in the steady-state, the harvest and growth in a single time period would be $\bar{g}$. With this level of resource stock and $\bar{\tau}$ acres, $\bar{g}$ represents a harvest of $\bar{V}$ units of volume per time period.

The Wicksell-Böhm-Baverk solution occurs at that τ where the per-cent growth rate is equal to the interest rate. This solution was found under the assumption that the entrepreneur maximizes net revenues for a single rotation. The similar problem of maximizing net revenues for rotations in perpetuity was given by:

$$\underset{\tau,L}{\mathrm{MAX}}\left\{\frac{PF(L;\tau)e^{-\rho\tau} - \omega L}{1 - e^{-\rho\tau}}\right\} \qquad . . . (2.23)$$

The solution to equation (2.23) is denoted (L^*, τ^*) and was illustrated in Figure 2.2. In Figure 4.1 the solution to the problem indicates that an acre of forest land should be harvested at the age τ^*. Looking at equation (2.28), the Wicksell-Böhm-Baverk solution generally results in a longer rotation than given by (2.23). The two approaches yield the same solution when:

$$\frac{\partial V}{\partial L} \cdot \frac{L}{V} = 1 \qquad \ldots (4.14)$$

Smith's static model of Chapter Three indicated that the resource should be managed such that:

$$Q'(V) = 0 \qquad \ldots (4.15)$$

In Figure 4.1, this point is given in the south-west quadrant as V_s, g_s. Smith states that this indicates that the resource should be managed so as to maximize sustained yield. However, the true maximum sustained yield is given by the solution to (2.16). A discussion of this discrepancy will be postponed until the dynamic models have been reviewed as they are subject to the same problem.

Plourde's dynamic model yields the result:

$$\rho = Q'(V) \qquad \ldots (4.16)$$

In the steady-state therefore, the stock of the resource is determined by (4.16). In the south-west quadrant of Figure 4.1, the stock of the resource is given by V_p and the growth in a single time period is g_p. This result indicates that a harvest of g_p in each time period will leave the stock of the resource unchanged.

Both Brown's dynamic model and the dynamic model based on Smith's assumptions essentially indicate a similar conclusion. The results are not unambiguous however. For the purposes of comparison with the static models, (4.16) will be used as a general result from the dynamic approach although the reader is reminded that such a result is available only under additional assumptions concerning the co-state variable.

At this point, the discrepancy between Smith's maximum sustained yield and the true maximum sustained yield given by equation (2.16) should be discussed. Smith's notion of maximum sustained yield, given by V_s with a growth and harvest of g_s is not possible in reality. His notion indicates that an acre of forest land could be grown until age τ_s and in each succeeding time period the growth g_s harvested. This is not possible. The growth in each period of time is added to the resource stock and once added is inseparable. For example, the yearly growth of a tree cannot be separated from the tree. The whole tree must be harvested or the whole acre harvested. All of the

static models of Chapter Two recognize this point.

This problem is not unique to the use of dynamic models in forestry. Indeed, in the case of fisheries one is unable to remove just the yearly growth from a fish or the yearly growth from a fishing ground. What then is the value of the dynamic approach in the determination of the steady state stock of the resource? The results may be completely discounted as indicating nothing or it may be said that the approach *implies* something. It may correctly be said that the Smith static model implies that the resource should be managed to maximize sustained yield. The Plourde result therefore implies that the resource should be managed not at maximum sustained yield but harvested at sustained level at some earlier age.

For the case of forestry, one approach that may possibly reconcile this difficulty is to define the resource constraint in the dynamic models as:

$$\frac{dV}{dt} = R(V) - C \qquad \text{. . . (4.17)}$$

In terms of (2.15) and therefore assuming $L^* = 0$:

$$R(V) = \frac{(\alpha - \log_e V)\cdot V}{\beta} = \frac{V}{\tau} \qquad \text{. . . (4.18)}$$

If (4.17) were used as the resource constraint, the result

of the Smith model (static) would truly be maximum sustained yield since:

$$R'(V) = 0 \qquad \ldots (4.19)$$

yields the same solution as:

$$\underset{\tau}{\mathrm{MAX}}\left\{\frac{V}{\tau}\right\} \qquad \ldots (4.20)$$

and (4.20) does yield maximum sustained yield. In Figure 4.1 the Smith solution would coincide with $\bar{\tau}$. In the Plourde model, the steady-state level of the resource stock would then be given by:

$$\rho = R'(V) \qquad \ldots (4.21)$$

and as long as $\rho > 0$ this would indicate $\tau_p < \bar{\tau}$. According to (4.17) with $(dV/dt) = 0$, in each time period the harvest would be given by:

$$C = R(V) = \frac{F(L^*;\tau_p)}{\tau_p} \qquad \ldots (4.22)$$

Therefore $1/\tau_p$ acre would be harvested in each period.

In this section the resource constraint was given by the growth function for one acre. If for example, a

forest of τ_p acres were considered, would the result of the Plourde model be:

$$C = F(L^*, \tau_p) \qquad \ldots (4.23)$$

in the steady state? Intuitively this would follow from (4.22) but from a mathematical standpoint problems develope.

An Aggregation Problem

In this section, a problem of aggregation that concerns the dynamic models will be shown. No general solution to this problem exists, however under certain forms of the growth function the problem does not arise.

The reader will recall that the production function for forestry is given by:

$$\bar{V} = g(a, b, 0, \tau) \qquad \ldots (2.11)$$

where the assumption of homogeneity in land (b) and labor (a) services allowed (2.11) to be written:

$$V = F(L; \tau) \qquad \ldots (2.13)$$

In fact, the units in which land services are measured and

the measurement of τ are very much bound together. For example, suppose a forest is composed of 2 acres, both the same age ($\tau_1 = \tau_2$). For this hypothetical forest, the aggregate growth function (V_f) would be:

$$V_f = 2V = 2F(L;\tau_1) = 2F(L;\tau_2) \quad . . . (4.24)$$

To illustrate this point further we may take as an example the growth function:

$$\bar{V} = e^{(\alpha-\beta/\tau)} A\left\{\delta a^{-z} + (1 - \delta)b^{-z}\right\}^{-\frac{1}{z}} \quad . . . (4.25)$$

where $A = 1/(1 - \delta)^{-\frac{1}{z}}$. For a = 0, (4.25) simplifies to:

$$\bar{V} = e^{(\alpha-\beta/\tau)} \cdot b \quad . . . (4.26)$$

For the two acres τ_1, τ_2, (4.24) becomes

$$\bar{V} = V_f = 2e^{(\alpha-\beta/\tau_1)} = 2e^{(\alpha-\beta/\tau_2)} \quad . . . (4.27)$$

If, however, the two acres are of different ages, (4.26) is no longer true. Obviously the choice of the units of b and the determination of τ are bound together.

The choice of the acre as the unit of measurement of b is not arbitrary. In choosing such a measure, the homogeneity of the trees composing the acre is important. This is not a novel problem. Conceptually, a

unit of capital may be a machine. The machine is however composed of various pieces and if all the pieces are essentially the same age or vintage there may be little dispute over this conceptual unit of measurement. Another factor contributing to the choice of the machine as an ideal unit of measurement is the fact that at some point in time the machine will be totally replaced. The choice of the acre as the unit of measurement may be the ideal unit since an acre may usually be composed of trees homogeneous with respect to age and the acre may be the smallest practical unit of harvest.

If the forest is composed of n acres and the age distribution of the forest is given by:

$$nD(\tau) \qquad \qquad . \; . \; . \; (4.28)$$

where $D(\tau)$ is the probability density function for the forest, then the aggregate volume function is given by:

$$V_f = n\int_0^\infty D(\tau)e^{(\alpha-\beta/\tau)}d\tau \qquad \qquad . \; . \; . \; (4.29)$$

It is quite obvious from (4.29) that the age distribution and the aggregate volume function are very much tied together. In the dynamic models, where such an aggregate function forms the resource constraint any harvest may change the form of (4.29).

There are cases in which the aggregate function will not change form with a harvest. An example of such a case is:

$$V = A\left\{1 - e^{-\beta\tau}\right\} \qquad . . . (4.30)$$

Equation (4.29) therefore becomes:

$$V_f = n\int_0^\infty D(\tau)A(1 - e^{-\beta\tau})d\tau \qquad . . . (4.31)$$

which can be immediately simplified:

$$V_f = nA\left\{\int_0^\infty D(\tau)d\tau - \int_0^\infty D(\tau)e^{-\beta\tau}d\tau\right\} \qquad . . . (4.33)$$

Noting that:

$$\int_0^\infty D(\tau)d\tau = 1, \qquad . . . (4.34)$$

$$V_f = nA\left\{1 - \int_0^\infty D(\tau)e^{-\beta\tau}d\tau\right\} \qquad . . . (4.35)$$

The integral in (4.35) may be approximated by $e^{-\beta\tau_f}$.

Therefore:

$$V_f = nA\left\{1 - e^{-\beta\tau_f}\right\} \qquad . . . (4.36)$$

There are two properties which must hold for meaningful aggregation. Having shown the effect of the age distribution above, it will be convenient to show that the two properties hold for (4.36) in discrete notation. The first property is simply (4.31) or:

$$V_f = \sum_{i=1}^{n} A(1 - e^{-\beta\tau_i}) \qquad . . . (4.37)$$

This requires the sum of the volumes of all n acres be equal to the volume given by the aggregate relationship. The second property:

$$\frac{dV}{d\tau_f} = \sum_{i=1}^{n} \frac{d[A(1 - e^{-\beta\tau_i})]}{d\tau_i} \qquad . . . (4.38)$$

Equation (4.38) requires that the growth rate given by the aggregate equation (4.36) be equal to the sum of the growth rates of each acre in the forest. Both conditions are satisfied for:

$$\tau_f = \frac{1}{-\beta} \log_e \sum_{i=1}^{n} e^{-\beta\tau_i} \qquad . . . (4.39)$$

Aggregation of this type is not possible for a sigmoid growth function. It may be possible to find an

aggregate form that satisfies the first property but invariably the second property is violated. The inflection point causes these problems.

This aggregation is not consistent.[2] Knowledge of V_f will not allow the determination of the V_i. Therefore, for any V_f a variety of possible age distributions may exist. However, for all the age distributions the aggregate form remains the same.

Smith seems to be aware of the possible effect of the control variable on the growth function and therefore states:

> In (2.2) we assume no interaction between the total harvest and the growth properties of the population mass . . . .[3]

Smith's equation (2.2) is the same as (3.32) in all but notation.

Concluding Remarks

To conclude a study of this nature in a satisfactory manner requires a reiteration of the general problem forest managers face. Presented initially with a forest composed of many acres of different ages the manager may first determine what the optimal configuration of the forest might be. The scarcity of wood could lead the traditional forester to the conclusion that the maximization of sustained yield is the appropriate goal. The rotation length and the nature of the steady state forest are there-

fore determined. Once these decisions have been made the forest manager must then decide how, given the initial state of the forest, to achieve the configuration of the steady state forest. Traditionally, market prices play no role in this process.

It is interesting to note that the management of forests in this case begins with a static problem and solution and then, given the initial state of the forest and desired state, the dynamic problem is solved. Depending upon the approach taken in the dynamic problem, the maximum sustained yield forest is achieved in a finite time or in the limit. The important point to recognize is that the steady states of the dynamic and static problems are the same.

The second chapter considered the static problem from both an economic and traditional forestry standpoint. The rotation lengths were determined and the nature of the steady state forest was discussed. The dynamic models of the third chapter examined essentially the same problem. No attention was given to the transition between the initial and final or steady state as such an analysis would require an explicit solution in each model. The steady states indicated by the economic static and dynamic problems are not the same.

The static analysis allows several conclusions to be drawn. The Böhm-Baverk--Wicksell formulation of the

problem leads to a longer rotation length than the Faustman approach of a perpetual series of rotations. In general it would be expected that the foresters' rotation length would be longer than that indicated by Böhm-Baverk or Wicksell. An unambiguous result is not possible. Comparative static analysis indicates that an increase in the rate of interest will lead to a longer, less intensive rotation if the entrepreneur maximizes discounted net revenues for a perpetual series of rotations. An increase in the price of timber will result in a shorter, more intensive rotation while an increase in the wage rate leads to a longer, less intensive rotation.

The dynamic models of the third chapter all view the resource as a mass or stock from which a flow of growth arises. It is further assumed that this growth can be harvested with no effect on the resource stock. In fact, the growth and the stock are inseparable. It is not possible to harvest the growth without also harvesting the stock associated with the growth. In these models, for any given resource stock, the growth is given by the first derivative of the volume function. This is correct if the flow and stock are separable. Since the stock associated with the growth is in reality harvested, the average volume curve gives all possible steady states. In the preceding pages of this chapter an alternative specification of the resource constraint which would reflect this fact was con-

sidered. However, the dynamic models incorporating such a change are still not correct.

The forest was considered as an aggregate of acres, each acre composed of trees of the same age. Each acre is therefore of a particular age and the harvest of an acre removes all the timber from that acre. The growth of the whole forest is obviously the sum of the growth of all acres and therefore the ages or age distribution of the forest is important to the volume function of the forest. The form of the aggregate volume function depends upon the age distribution. The harvest of one "old" acre with a given volume or the harvest of two "young" acres which, in total, comprise the same volume is not equivalent

The results of the dynamic models of chapter three should therefore be qualified. They are correct only if the resource stock and growth are separable and the control variable does not influence the form of the growth curve. For such a resource, the steady state occurs at some stock of the resource smaller than that stock at which growth is maximum. For any properly specified model of resource exploitation the initial price imputed to a unit of the resource stock, correctly determined, will lead to the achievement of the steady state.

The formulation of a realistic dynamic model of forest management should reflect the biological characteristics of the resource. The behavioral or policy implica-

tions of an economic model constructed upon unrealistic biological assumptions are open to question. The economist cannot reasonably expect the resource manager to accept the results of economic analysis unless the biological model is rigorous.

A properly specified economic model of forest management would attract considerable attention in both practice and theory. The determination of harvesting schedules for public forests and land is a complex task. Resource managers faced with this decision are gradually introducing economic criteria into the process. Such a model would undoubtably require considerable mathematical sophistication however the benefits of such effort would be very tangible.

Footnotes

[1]Specifically, the first derivative is easily expressed in terms of the resource mass (N). See footnote 10, Chapter Three.

[2]I am using the term consistent in the same context as: Green, H. A., *Aggregation in Economic Analysis* (Princeton University Press, 1964), p. 20.

The purpose of the passage in the text on this problem was only to demonstrate that for some forms of the growth function an aggregate form may be available which is not influenced by the control variable. If the individual growth functions for each acre are the same and known with certainty then the aggregation, although *ad hoc*, is meaningful in relation to the properties which must be preserved.

[3]Smith, V., "Economics of Production from Natural Resources," p. 411.

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